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FUNDAMENTAL PRINCIPLES GOVERNING THE PATTERNING OF POLYHEDRA

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ABSTRACT:

This paper is concerned with the regular patterning (or tiling) of the five regular polyhedra (known as the Platonic solids). The symmetries of the seventeen classes of regularly repeating patterns are considered, and those pattern classes that are capable of tiling each solid are identified. Based largely on considering the symmetry characteristics of both the pattern and the solid, a first step is made towards generating a series of rules governing the regular tiling of three-dimensional objects.

Key words: symmetry, tilings, polyhedra

1. INTRODUCTION

A polyhedron has been defined by Coxeter as "a finite, connected set of plane polygons, such that every side of each polygon belongs also to just one other polygon, with the provision that the polygons surrounding each vertex form a single circuit" (Coxeter, 1948, p.4). The polygons that join to form polyhedra are called faces, these faces meet at edges, and edges come together at vertices. The polyhedron forms a single closed surface, dissecting space into two regions, the interior, which is finite, and the exterior that is infinite (Coxeter, 1948, p.5).

The regularity of polyhedra involves regular faces, equally surrounded vertices and equal solid angles (Coxeter, 1948, p.16). Under these conditions, there are nine regular polyhedra, five being the convex Platonic solids and four being the concave Kepler-Poinsot solids. The term regular polyhedron is often used to refer only to the Platonic solids (Cromwell, 1997, p.53). There are only five possible regular convex polyhedra (or Platonic solids) composed of regular polygonal faces with all vertices equally surrounded: the (regular) tetrahedron, (regular) octahedron, cube (or regular hexahedron), (regular) icosahedron and the (regular) dodecahedron (Figure 1). The proof of the existence of only five is discussed in Euclid's Elements, Book XIII (1956, proposition 18).

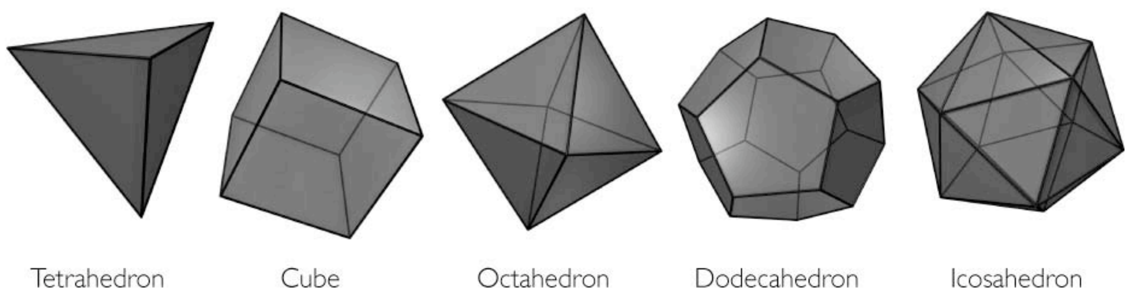

Figure 1: The regular polyhedra (or Platonic solids)

The regular polyhedra, and other more complex solids, have played an important role in art, design, engineering and science down through the centuries. In the Timaeus, Plato announced that the triangle (or the tetrakyst) was the building block of the universe. He presented this rather fanciful idea and others about creation, suggesting a process not unlike a geometric progression. Also in the Timaeus, Plato described how triangles made up the five regular solids and how these were associated with the four elements (earth, fire, air and water) and the universe (Annas, 1996, pp.1190–1193). Polyhedra have been associated with art and design for many centuries. A notable peak was reached during the Italian Renaissance. Relevant artists include Paola Uccello (1397–1475), Piero della Francesca (1410–1492), Leonardo da Vinci (1452–1519) and Luca Pacioli (1445–1517). Fra Giovanni da Verona produced a collection of intarsia (mosaics of inlaid wood) depicting various polyhedra around 1520. Interest was shown also by Albrecht Durer (1471–1528) and M. C. Escher (1898–1972). Johannes Kepler (1571–1630) produced some early drawings.

As a component of a wider ranging research project (covering many of the geometric aspects of design) the authors are currently considering the symmetry rules that may govern the regular patterning (or tiling) of the five regular polyhedra. The important challenge is to ensure precise registration and the absence of gaps or overlaps; this is neither a trivial nor a straightforward matter. Readers who doubt this may wish to reflect on the tiling problems associated recently with the space shuttle Discovery.

When regular repeating patterns, which perform with satisfaction on the Euclidean plane, are folded several times, into different planes, their component parts will not correspond readily. Only certain pattern types, with particular symmetry characteristics, are suited to the precise patterning of each Platonic solid. There are various difficulties encountered in attempting to apply two-dimensional repeating designs to regular polyhedra, avoiding gap and overlap and ensuring precise registration. This paper acknowledges the symmetry characteristics of importance to the process and identifies appropriate pattern types for each of the regular solids. Relevant illustrations are provided. It should be noted that this paper reports on the outcome of research completed to date, and that it is recognized by the authors that the results presented are not all encompassing. It is the intention to develop further the systematic means by which particular patterns may be applied to polyhedra. The longer-term intention is to extend the enquiry to take account of more complex solids (such as, for example, the semi-regular polyhedra or Archimedean solids).

2. LATTICE STRUCTURES, FUNDAMENTAL REGIONS AND POLYHEDRAL NETS

It is well established that regularly repeating patterns (or tilings) exhibit symmetries, which combine to produce seventeen possibilities across the plane (shown schematically in Figure 2 together with the relevant internationally accepted notation from the International Union of Crystallography). Proof of the existence of only seventeen all-over pattern classes was provided by Weyl (1952), Coxeter (1969) and Martin (1982). Designs possessing the same symmetry combinations are said to belong to the same symmetry class, and may be classified accordingly. Further accounts of the classification and construction of regularly repeating patterns and tilings are been given by Woods (1935), Schattschneider (1978), Stevens (1984), Washburn and Crowe (1988) and Hann and Thomson (1992) and Hann (2003). In the context of this paper it is of importance to recognize that across the seventeen pattern types, different combinations of symmetry are exhibited (e.g. reflection axes in various directions and/or two-, three-, four- and six-fold rotational symmetry).

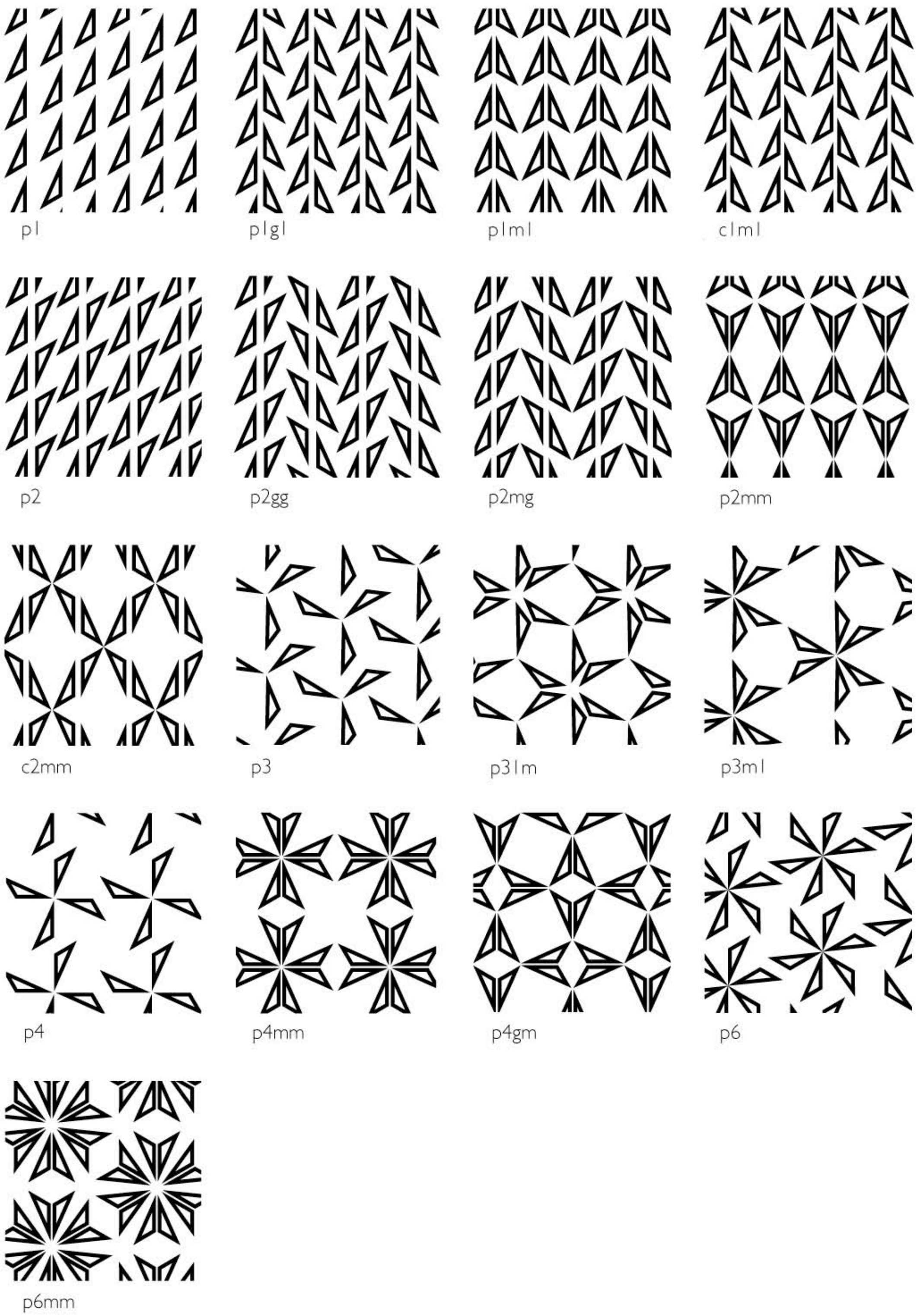

Figure 2: Schematic representations of the seventeen classes of regularly repeating patterns.

Turning attention to three-dimensions, a polyhedron is a three-dimensional solid that consists of a collection of polygons, usually joined at their edges. Composed of faces identical in size and shape, equally surrounded vertices and equal solid angles, the regular convex solids, known as the Platonic solids, are highly symmetrical polyhedra. The symmetry characteristics of reflection and rotation that govern the properties of regularly repeating patterns and tilings, are of importance also to three-dimensional solids.

In regularly repeating plane patterns, a further geometrical element of importance to pattern structure is the underlying framework or lattice, of which there are five distinct types: parallelogram, square, rectangular, hexagonal, rhombic centered-cell. Each lattice is comprised of unit cells of identical size, shape and content, which contain the essential repeating unit or element of the pattern or tiling, as well as the symmetry instructions for the pattern's construction. The unit cells associated with the seventeen pattern classes are illustrated in Figure 3.

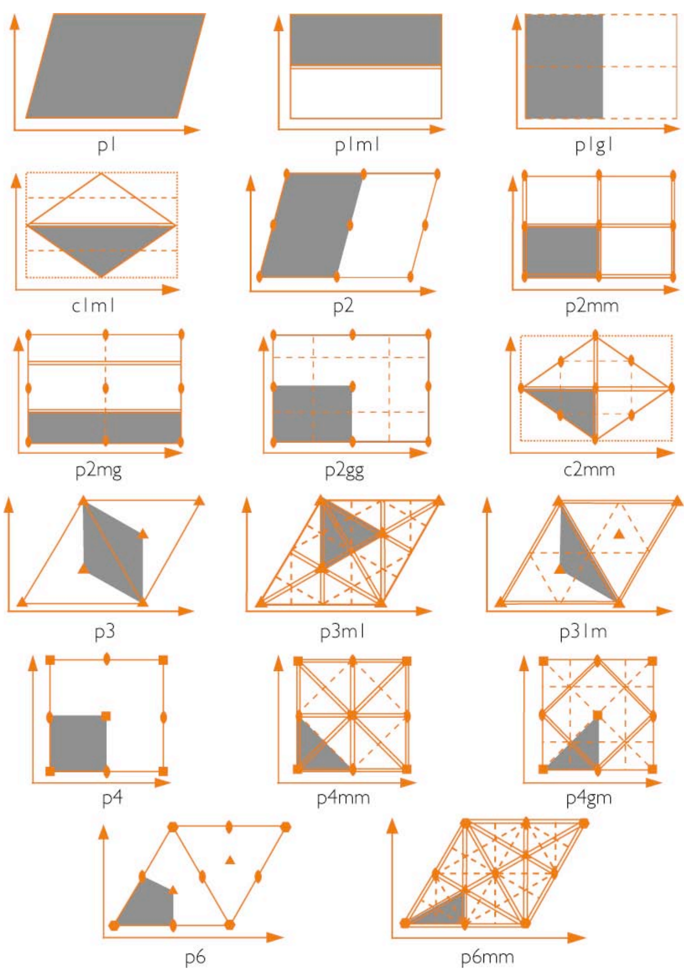

Figure 3: The seventeen unit cells associated with regularly repeating patterns.

The purpose of the investigation reported in this paper is to discover which of the seventeen pattern classes can repeat regularly around (or tile) the Platonic solids, applying only the restriction that the unit cell must repeat across the solid in exactly the same way that it does in the plane pattern.

The first step in regularly tiling a regular polyhedron involves matching the polyhedral faces to a suitable lattice type. Suitable patterns applicable to the tetrahedron, octahedron and icosahedron, whose faces are equilateral triangles, must be constructed on a hexagonal lattice, where the unit cell comprises two equilateral triangles. The cube is the only Platonic solid that is composed of square faces. All pattern classes constructed on a square lattice can thus be considered suitable to repeat around the surface of the cube. Patterning the dodecahedron, composed of regular pentagonal faces, requires a different approach. The regular pentagon, with five-fold rotational symmetry, cannot tile the plane without gap or overlap. There are, however, fourteen (known) types of equilateral pentagons that can tessellate the plane (Wells, 1991, pp.177-179). Probably the best known is the Cairo tessellation (Figure 4), formed by convex equilateral pentagons (equal-length sides, but different associated angles). Using the Cairo tessellation it is possible to devise a method by which to tile the dodecahedron.

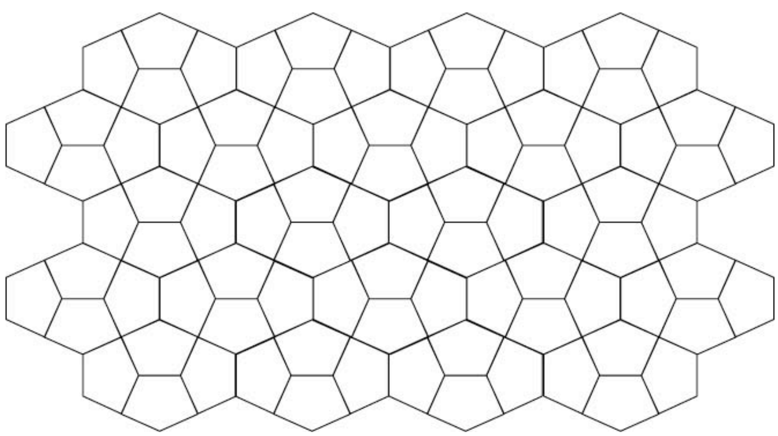

Figure 4: The Cairo tessellation.

3. TILING THE PLATONIC SOLIDS

Table 1 presents the findings of this research and lists the pattern classes appropriate to each of the Platonic solids and identifies the associated lattice structures, unit cell areas and other symmetry characteristics.

Platonic solid	Pattern class	Lattice structure	Area of unit cell on face	Vertices equivalent	Rotation present on vertices	Rotation present on edges	Rotation present on faces	Reflection present on edges	Reflection present on faces
Tetrahedron	p2	hexagonal	1/2	✔		2-			
	c2mm	hexagonal	1/2	✔		2-		✔	✔
	p6	hexagonal	1/2	✔	3-	2-	3-		
	p6mm	hexagonal	1/2	✔	3-	2-	3-	✔	✔
Octahedron	p3	hexagonal	1/2	✔	2-		3-		
	p31m	hexagonal	1/2	✔	2-		3-	✔	
	p3m1	hexagonal	1/2	✔	2-		3-		
	p3m1	hexagonal	1/6		a) 2- b) 2- c) 2-			✔	
	p6	hexagonal	1/2	✔	4-	2-	3-		
	p6	hexagonal	1/6		a) 4- b) 2-	2-			
	p6mm	hexagonal	1/2	✔	4-	2-	3-	✔	✔
	p6mm	hexagonal	1/6		a) 4- b) 2-	2-		✔	✔
Icosahedron	p6	hexagonal	1/2	✔	5-	2-	3-		
	p6mm	hexagonal	1/2	✔	5-	2-	3-	✔	✔
Cube	p4	square	1	✔	3-	2-	4-		
	p4mm	square	1	✔	3-	2-	4-	✔	✔
	p4gm	square	1	✔	3-	2-	4-		
Dodecahedron	p4	square	1/2	✔	3-	2-	5-		

Table 1: Symmetry characteristics of regularly tiled Platonic solids.

The tetrahedron, the octahedron and the icosahedron have, respectively, four, eight and twenty equilateral triangles as faces. The cube has six square faces and the dodecahedron has twelve regular pentagons as faces. Polyhedra may be drawn as plane diagrams, known as nets, in which the faces and edges of the polyhedron are shown. These may be considered as unfolded polyhedra. The net of a polyhedron may specify which edges are to be joined, as in some cases, it is possible for a net to represent several polyhedra. There may be several possible nets that represent a single polyhedron. The tiling of each of the five Platonic solids is considered below.

3.1. TILING THE TETRAHEDRON

The tetrahedron is a pyramid on a triangular base and the simplest of the Platonic solids (Coxeter, 1948, p.4). The tetrahedron consists of four equilateral triangular faces, four vertices and six edges. Each of the four vertices is surrounded by three triangles, which produces a *vertex angle* (sum of the vertex angles of the polygons meeting at each polyhedron vertex) of 180 degrees (Wenninger, 1971, p.1). The tetrahedron has seven axes of rotational symmetry: four axes of three-fold rotation connect each vertex with the midpoints of the opposite faces, and three axes of two-fold rotation pass through midpoints of the opposite edges. In addition to rotational symmetry, the tetrahedron possesses six planes of reflection passing through axes of two-fold rotation and the edges of the tetrahedron. The tetrahedron is its own dual polyhedron and therefore connecting the centers of the faces of a tetrahedron forms another tetrahedron.

It is possible to tile the tetrahedron with certain patterns possessing two-fold and six-fold rotational symmetry (classes p2, c2mm, p6 and p6mm). Relevant illustrations are given in Figures 5 to 12.

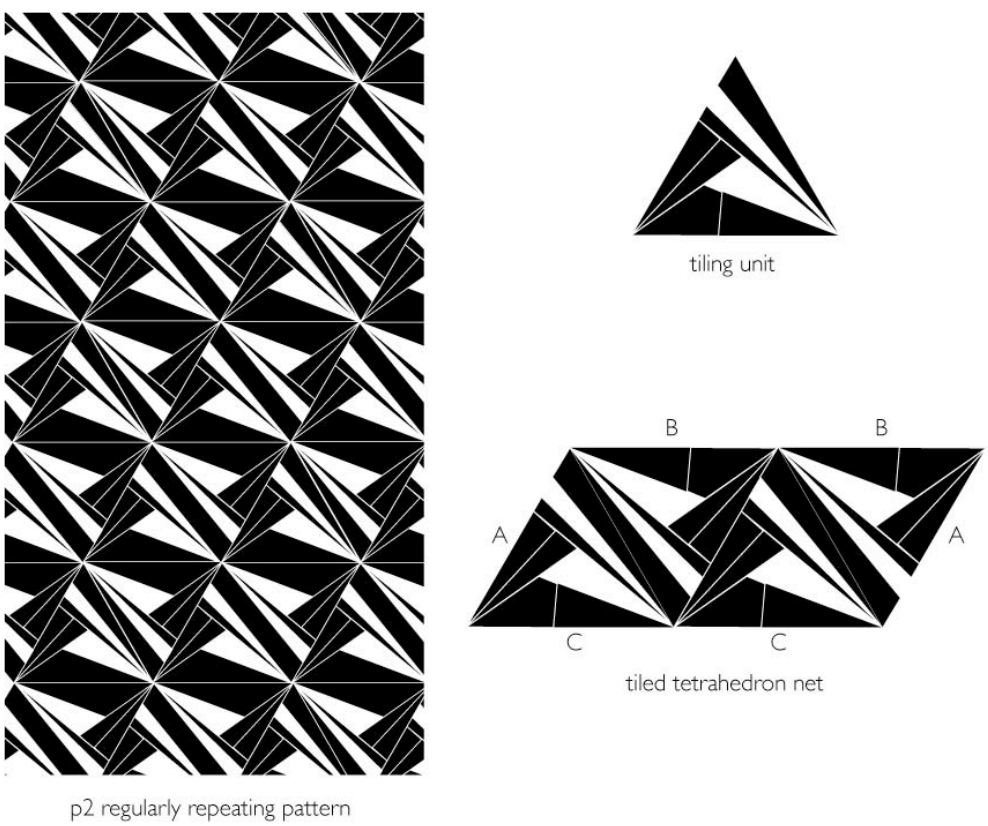

Figure 5: Illustration of a design for the tetrahedron regularly tiled with pattern class p2

Figure 6: The tetrahedron regularly tiled with pattern class p2

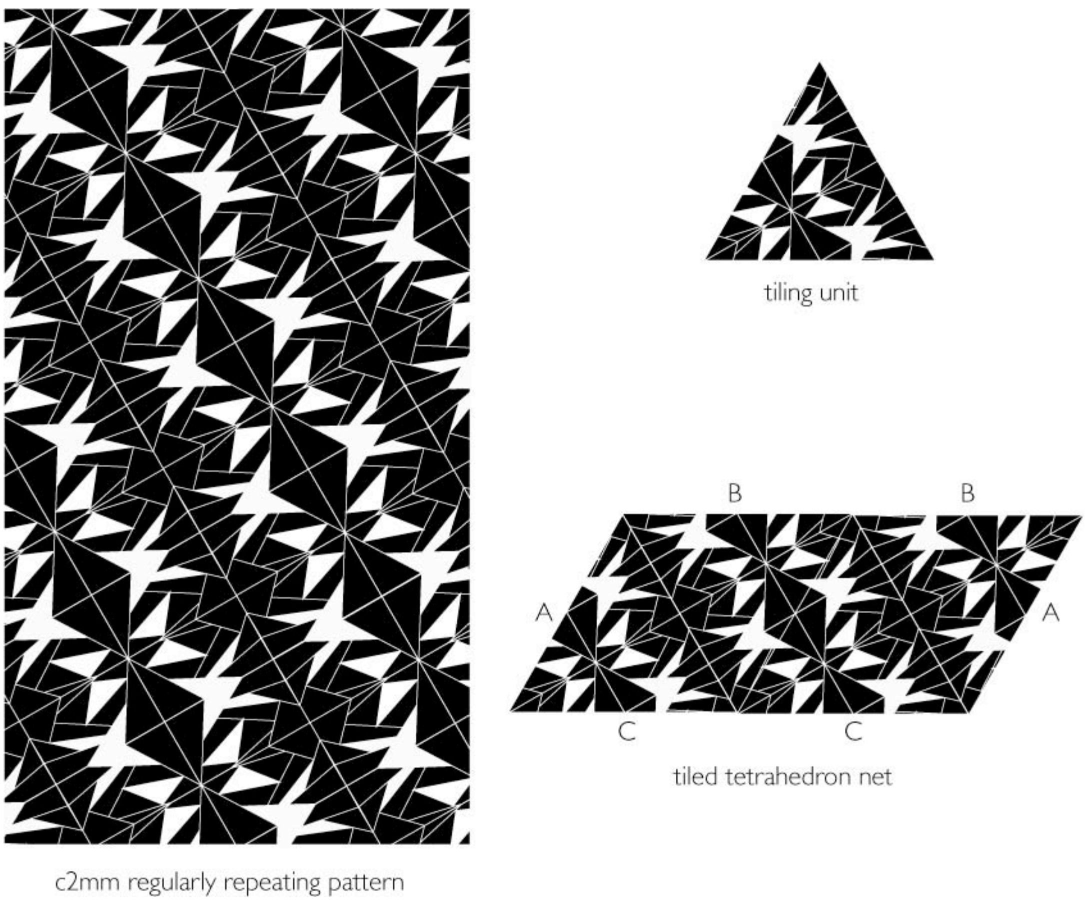

Figure 7: Illustration of a design for the tetrahedron regularly tiled with pattern class c2mm

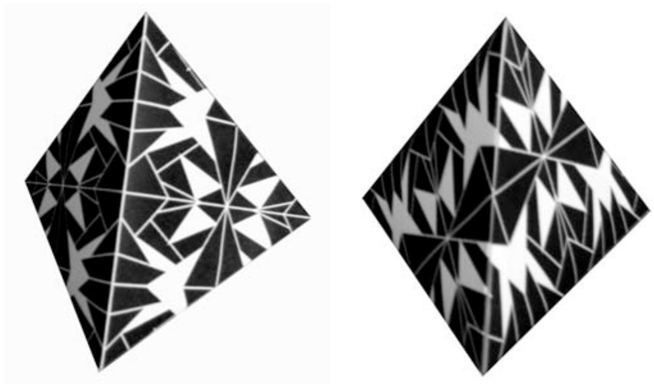

Figure 8: The tetrahedron regularly tiled with pattern class c2mm

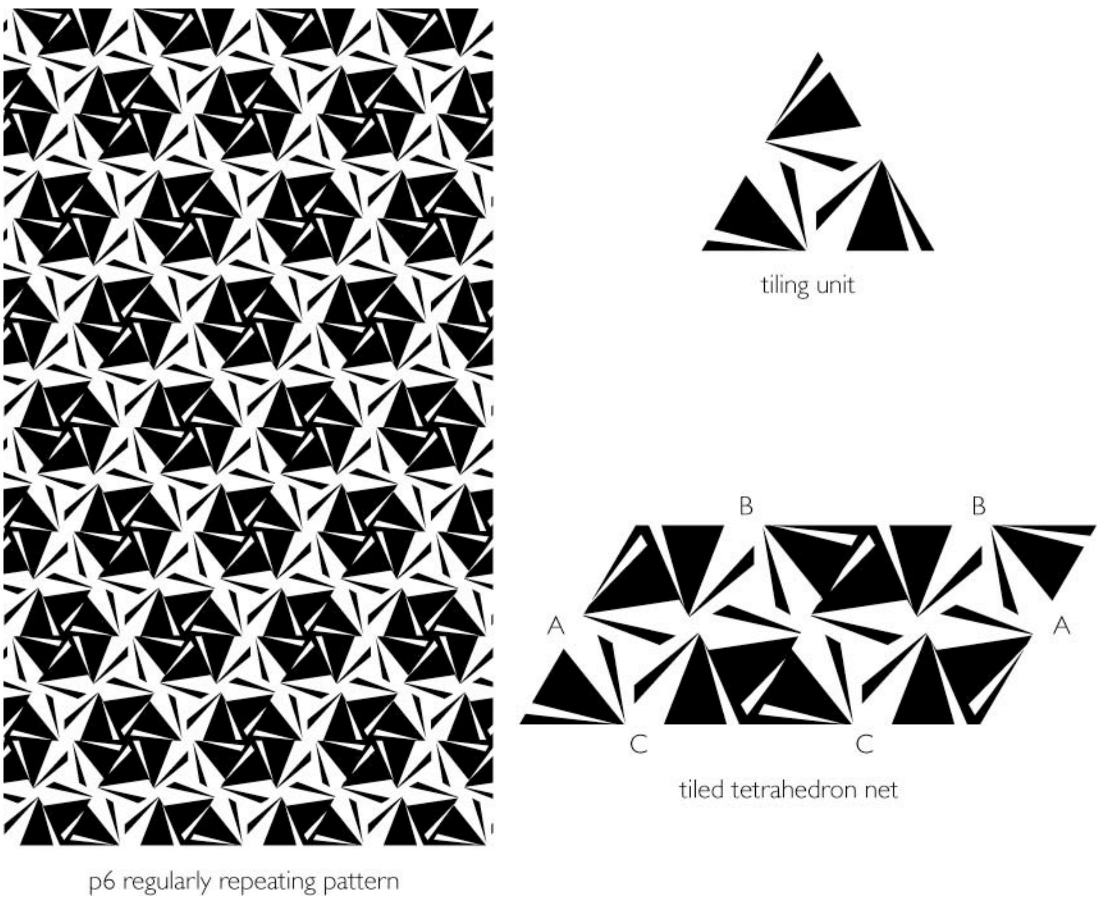

Figure 9: Illustration of a design for the tetrahedron regularly tiled with pattern class p6

Figure 10: The tetrahedron regularly tiled with pattern class p6

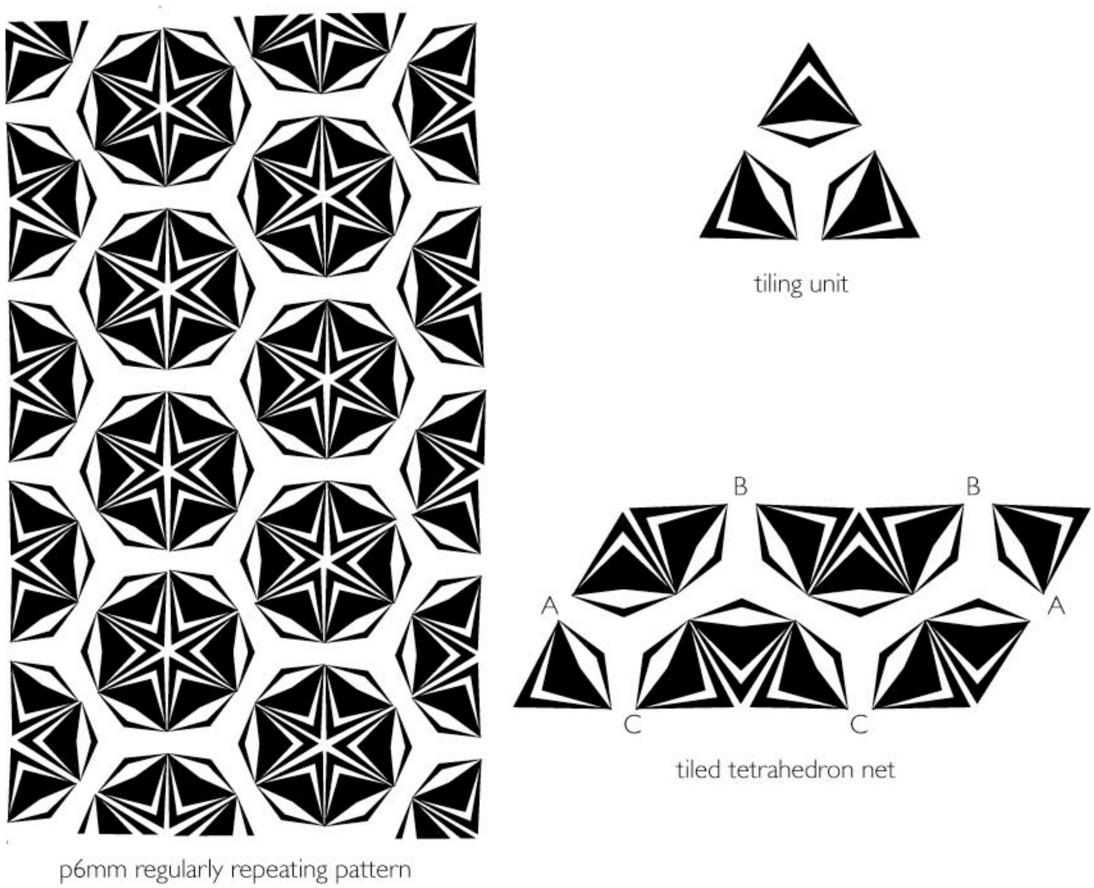

Figure 11: Illustration of a design for the tetrahedron regularly tiled with pattern class p6mm

Figure 12: The tetrahedron regularly tiled with pattern class p6mm

3.2. TILING THE OCTAHEDRON

The octahedron may be regarded as a dipyramid, formed by placing two equal square-based pyramids base to base [Coxeter, 1948, p.5]. The octahedron is composed of eight equilateral triangular faces, twelve edges and six vertices. Four triangular faces surround each vertex, which results in a vertex angle of 240 degrees [Wenninger, 1971, p.1]. The octahedron displays six axes of two-fold rotation passing through the midpoint of opposite edges, four axes of three-fold rotation connecting the centre of opposite faces and three axes of four-fold rotation joining opposite vertices. In addition to rotational symmetry, the octahedron exhibits nine planes of reflection. The octahedron possesses the same symmetry characteristics as the cube.

Pattern classes p3, p3m1, p31m, p6 and p6mm are each suited to patterning the octahedron. Relevant illustrations are provided in Figures 13 to 28.

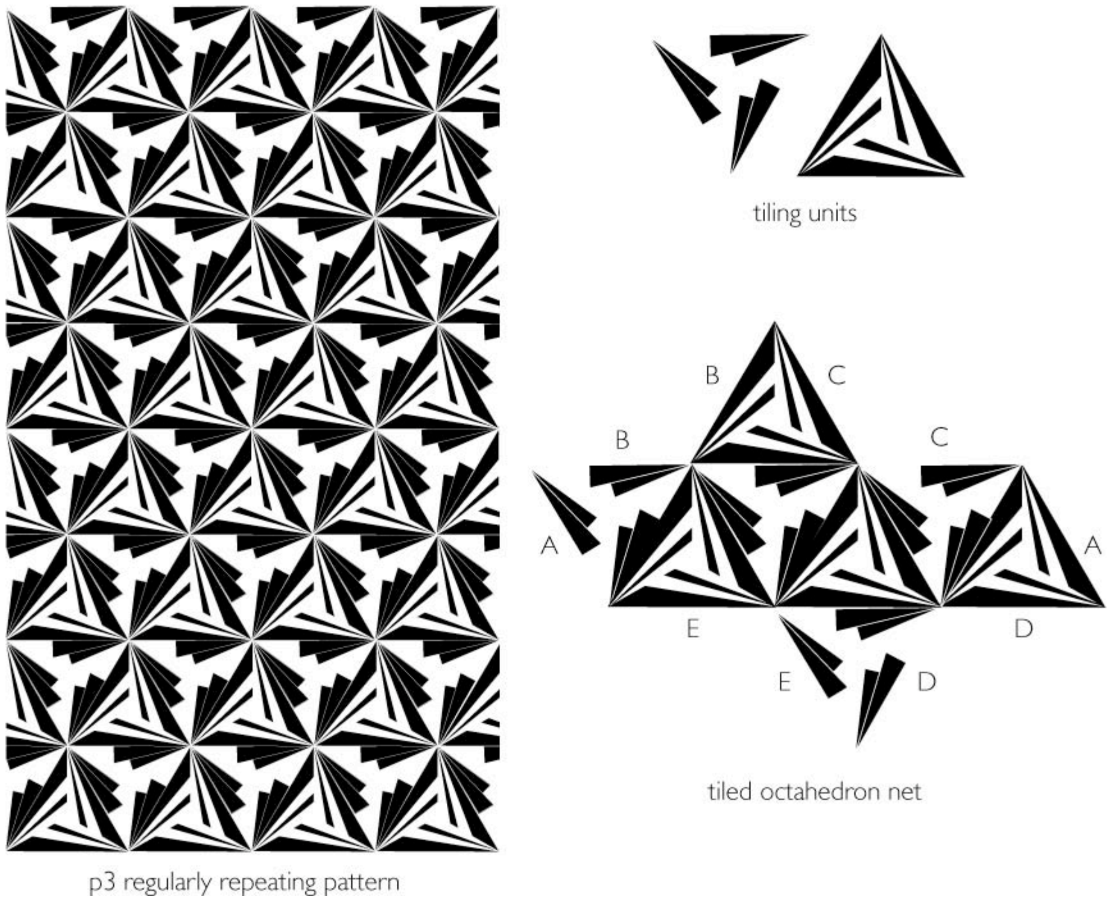

Figure 13: Illustration of a design for the octahedron regularly tiled with pattern class p3

Figure 14: The octahedron regularly tiled with pattern class p3

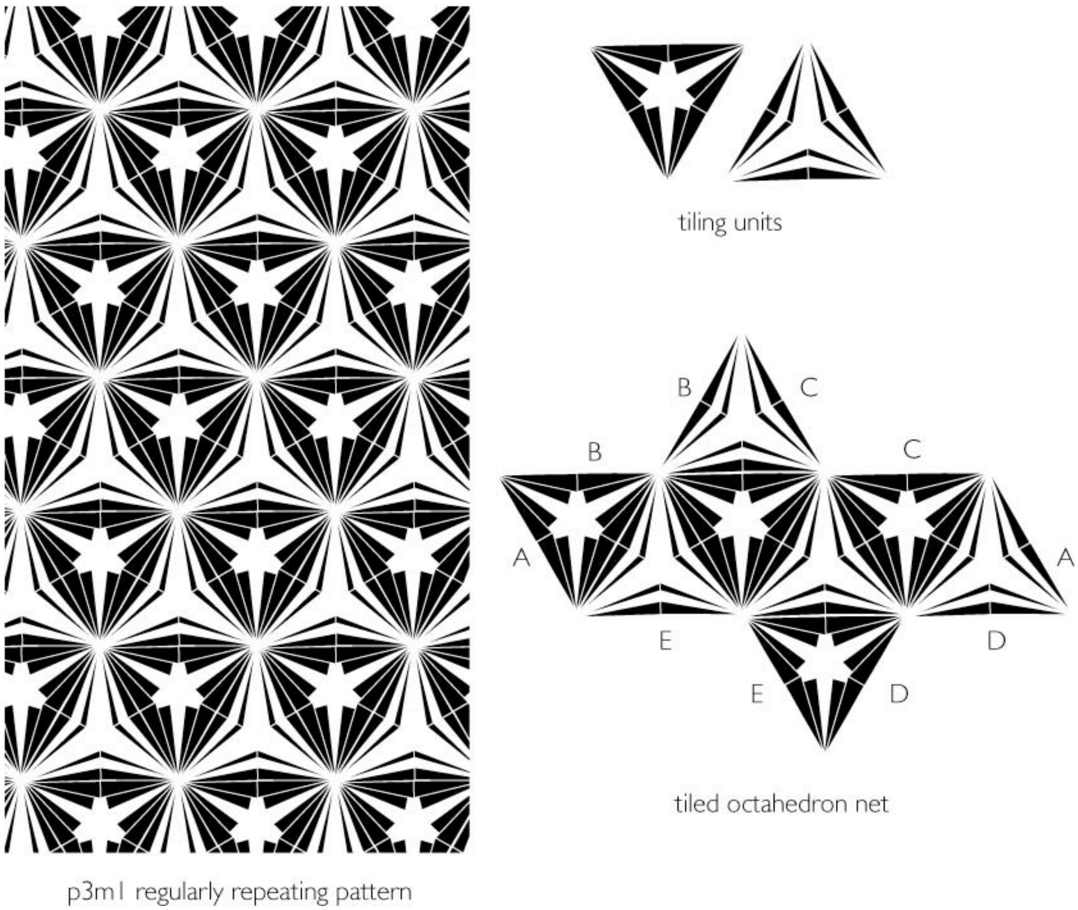

Figure 15: Illustration of a design for the octahedron regularly tiled with pattern class p3m1

Figure 16: The octahedron regularly tiled with pattern class p3m1

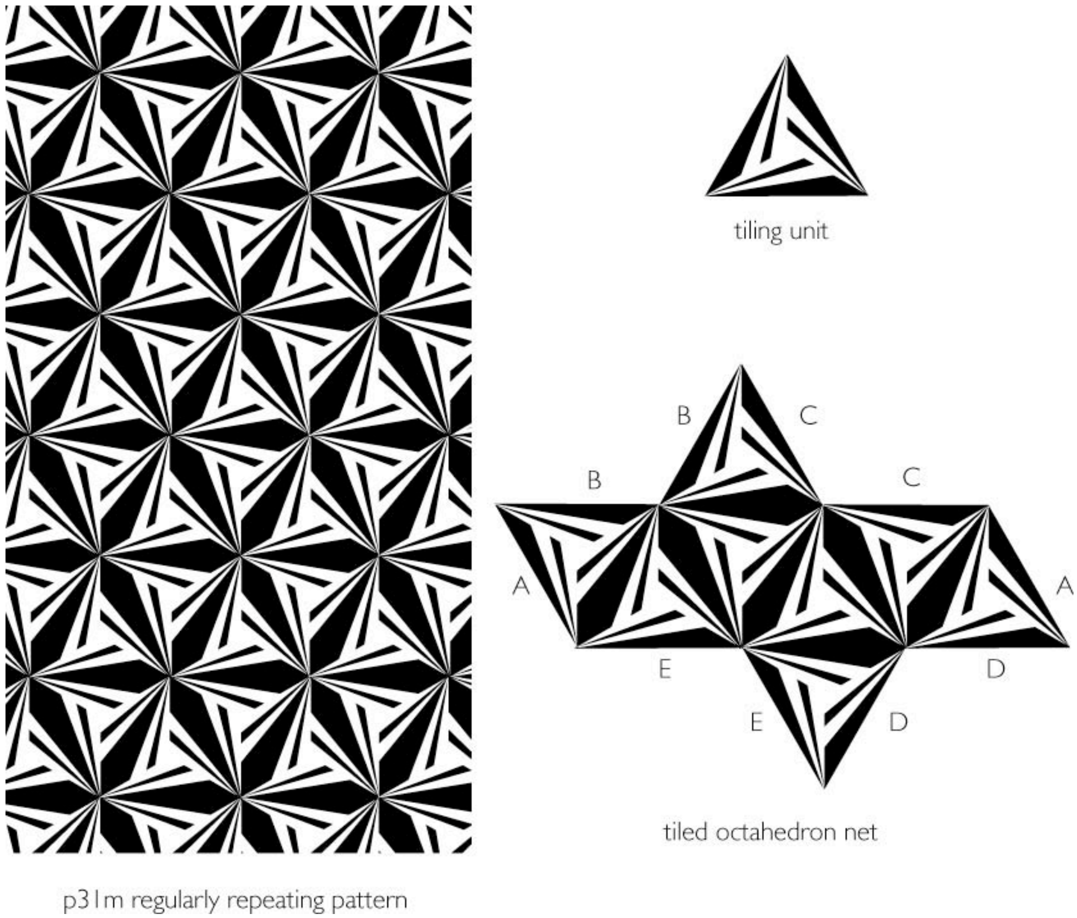

Figure 17: Illustration of a design for the octahedron regularly tiled with pattern class p31m

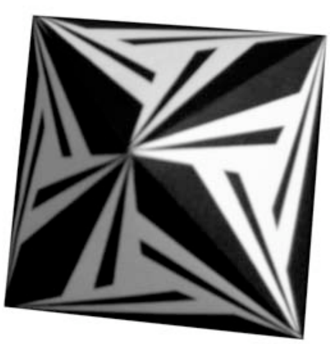

Figure 18: The octahedron regularly tiled with pattern class p31m

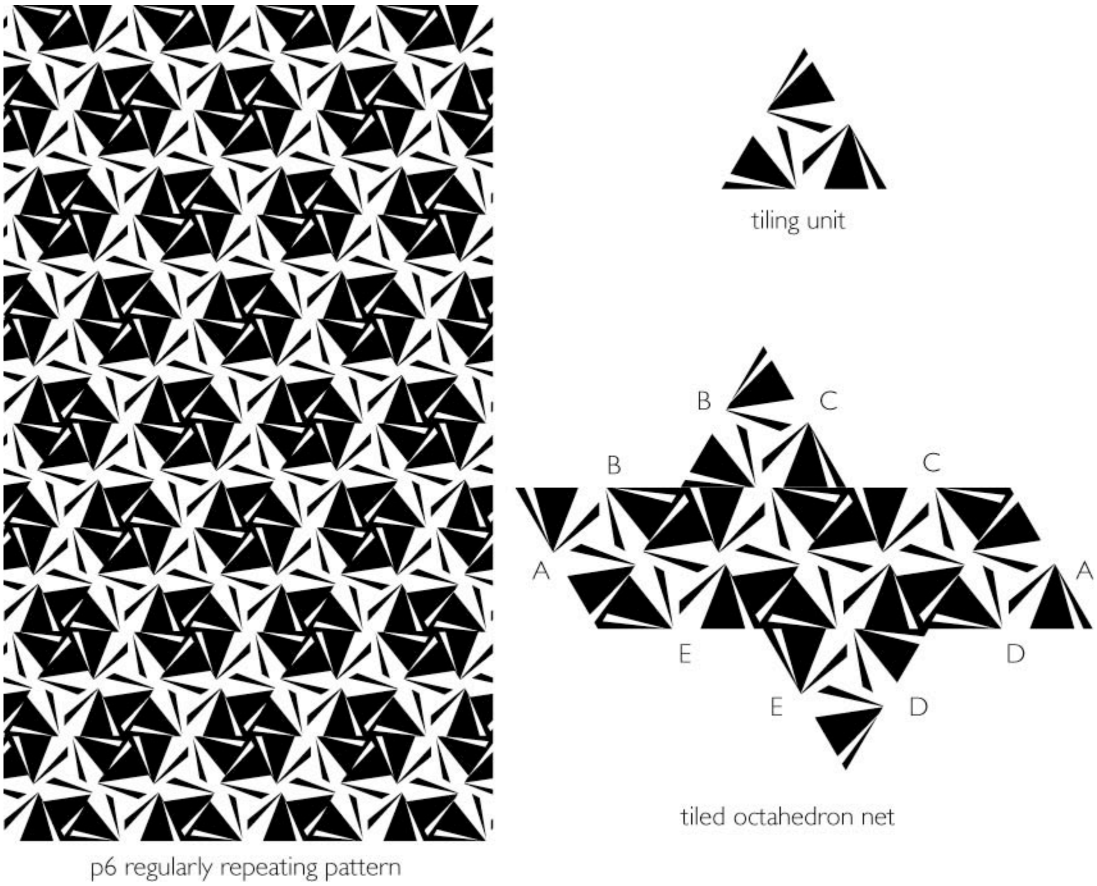

Figure 19: Illustration of a design for the octahedron regularly tiled with pattern class p6


Figure 20: The octahedron regularly tiled with pattern class p6

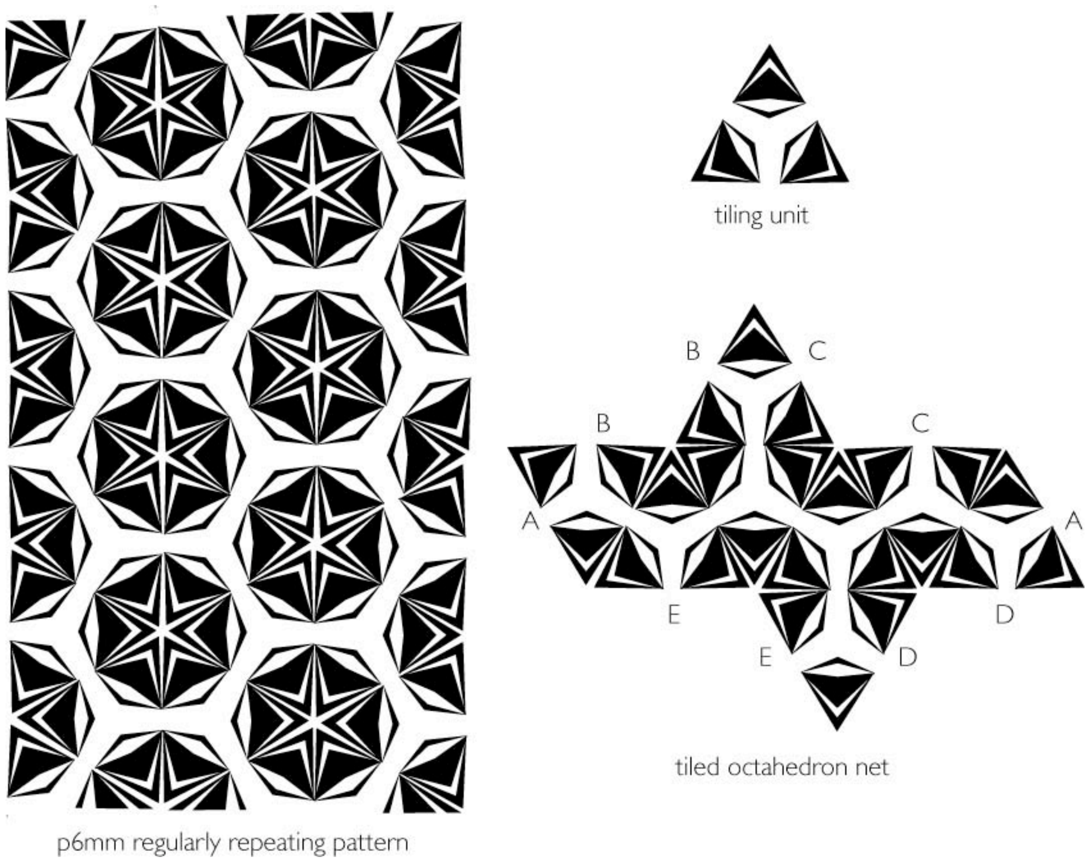

Figure 21: Illustration of a design for the octahedron regularly tiled with pattern class p6mm

Figure 22: The octahedron regularly tiled with pattern class p6mm

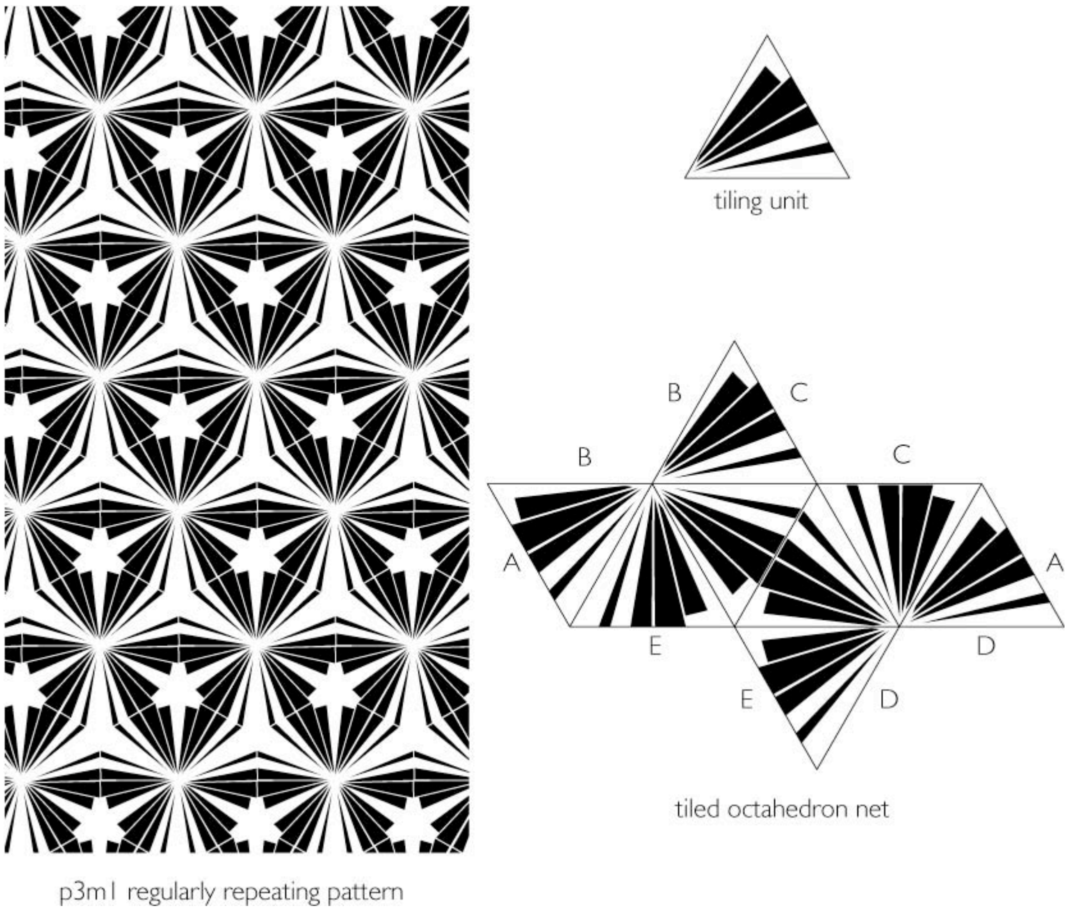

Figure 23: Illustration of a design for the octahedron regularly tiled with pattern class p3m1, where the area used to tile each face is equal to one-sixth of the unit cell

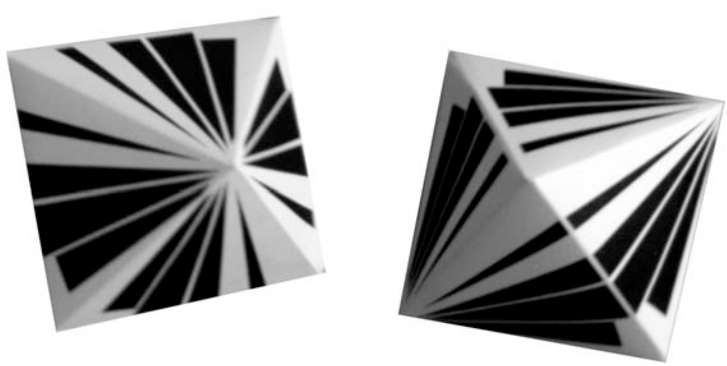

Figure 24: The octahedron regularly tiled with pattern class p3m1, where the area used to tile each face is equal to one-sixth of the unit cell

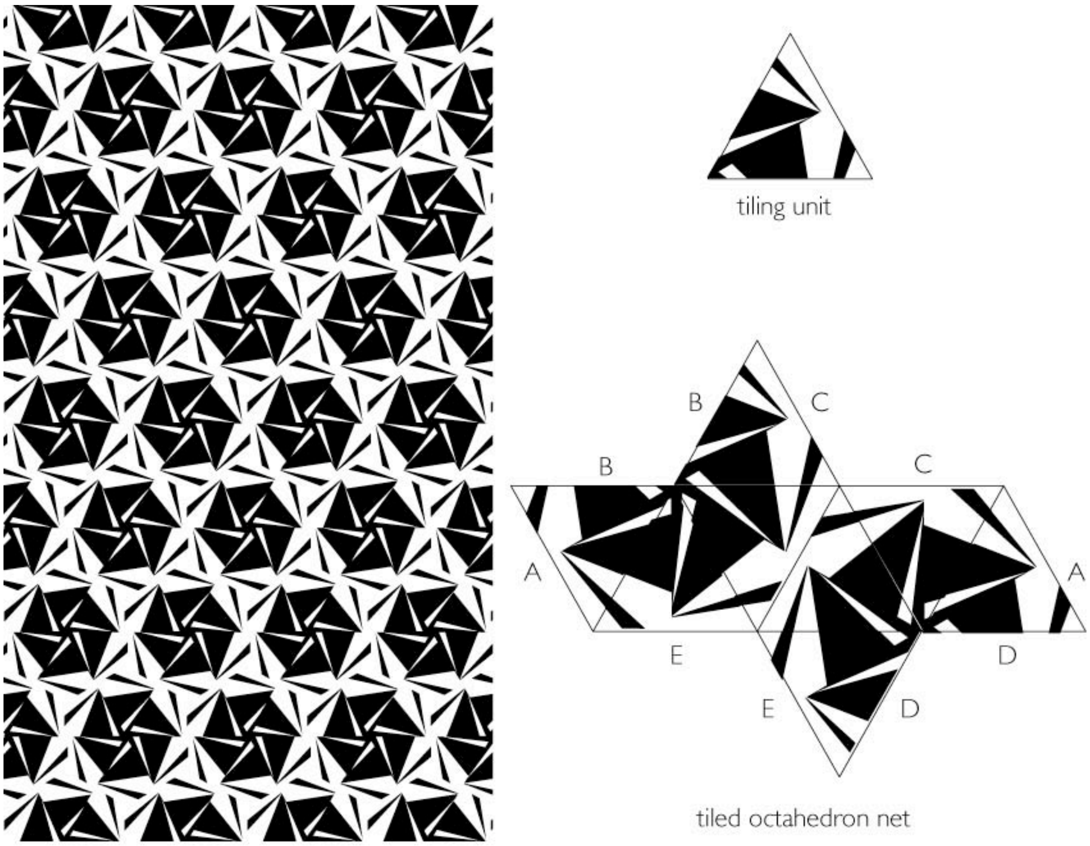

Figure 25: Illustration of a design for the octahedron regularly tiled with pattern class p6, where the area used to tile each face is equal to one-sixth of the unit cell

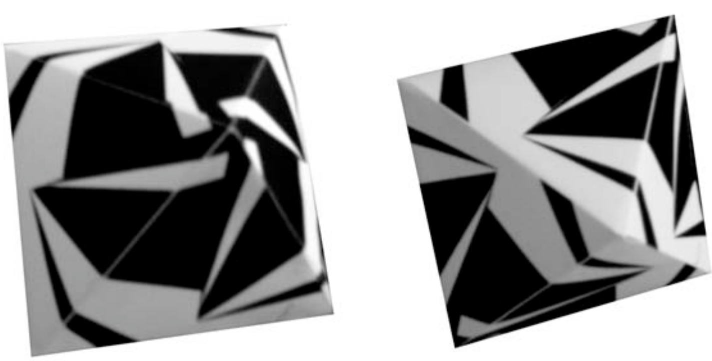

Figure 26: The octahedron regularly tiled with pattern class p6, where the area used to tile each face is equal to one-sixth of the unit cell

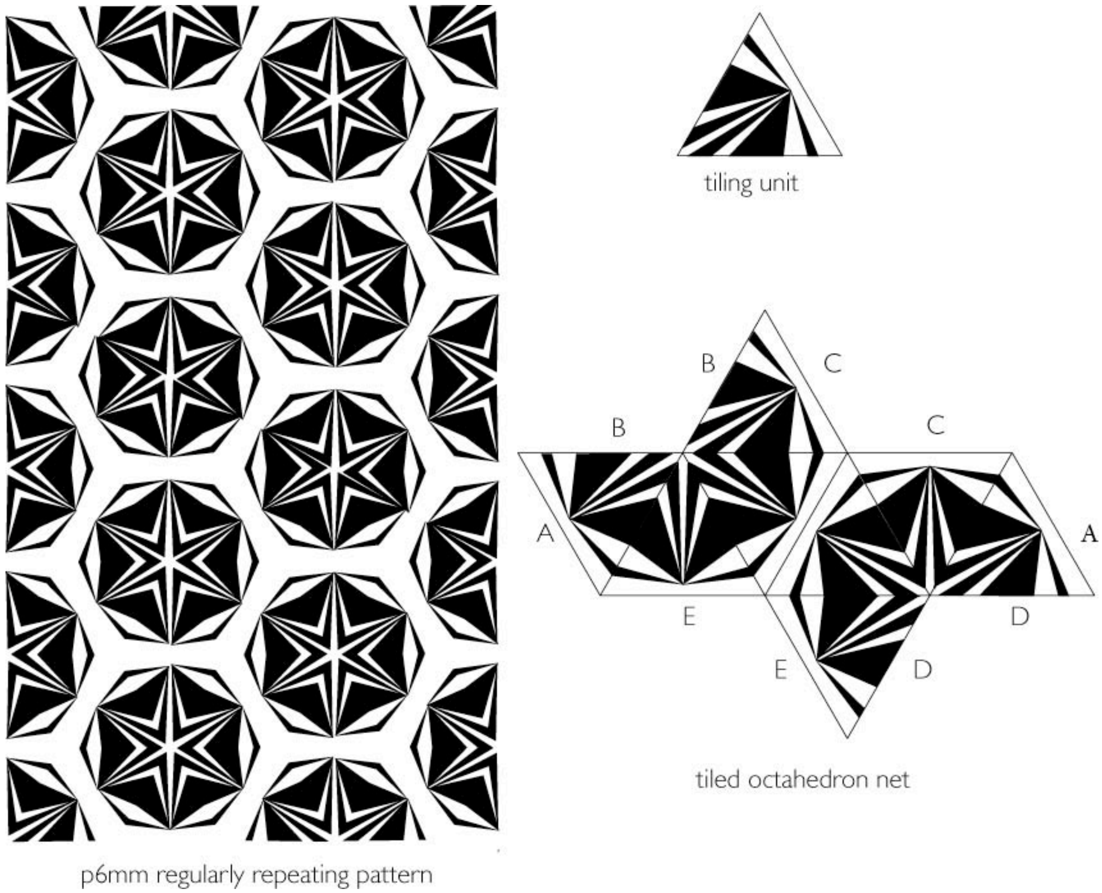

Figure 27: Illustration of a design for the octahedron regularly tiled with pattern class p6mm, where the area used to tile each face is equal to one-sixth of the unit cell

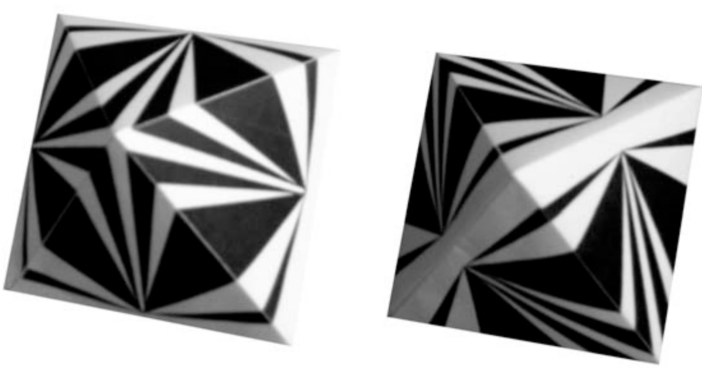

Figure 28: The octahedron regularly tiled with pattern class p6mm, where the area used to tile each face is equal to one-sixth of the unit cell

3.3.TILING THE ICOSAHEDRON

The icosahedron consists of twenty equilateral triangular faces. A total of twelve vertices and thirty edges are present on the solid. Each vertex is surrounded by five equilateral triangles producing a vertex angle of 300 degrees (Wenninger, 1971, p.1). The icosahedron displays a similar set of symmetries as the dodecahedron. Fifteen axes of two-fold rotation join the midpoints of opposite edges. Ten axes of three-fold rotation pass through the centers of opposite faces, and six axes of five-fold rotation connect opposite vertices. Comparable with the dodecahedron, the icosahedron also exhibits fifteen planes of reflection.

Only pattern classes containing six-fold rotation are applicable to regularly tiling the icosahedron. The icosahedron possesses the same rotational properties found in pattern class p6 and therefore the icosahedral rotational symmetries are preserved in the tiled solid. In addition to pattern class p6, with its inherent rotational symmetry, pattern class p6mm, which exhibition reflection also, accommodates the reflection properties of the icosahedron. Relevant illustrations are provided in Figures 29 to 32.

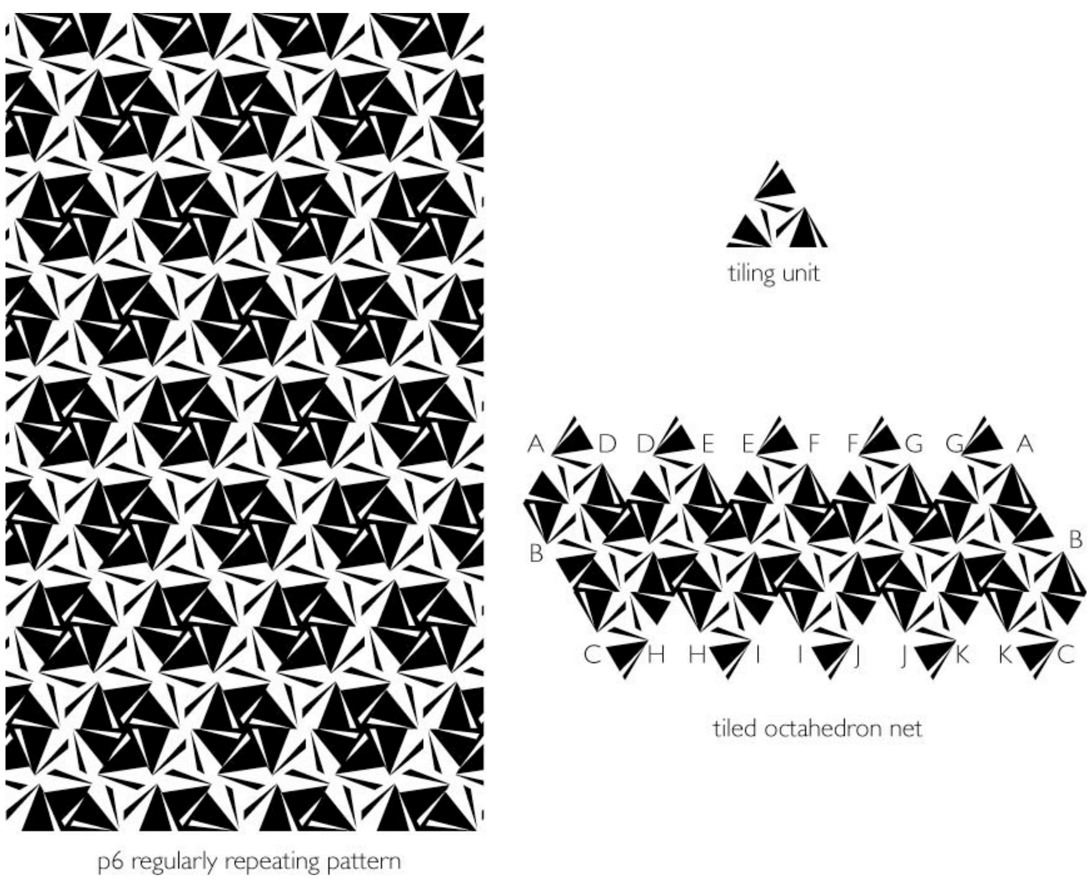

Figure 29: Illustration of a design for the icosahedron regularly tiled with pattern class p6

Figure 30: The icosahedron regularly tiled with pattern class p6

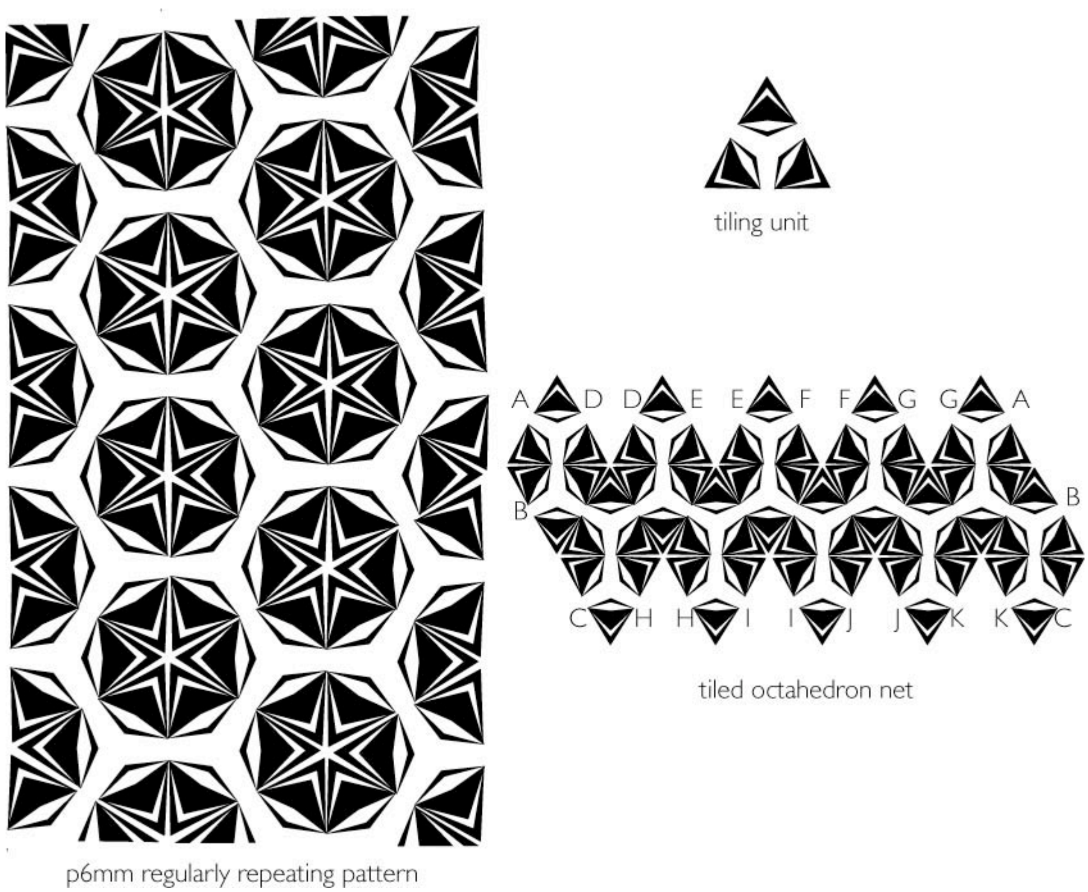

Figure 31: Illustration of a design for the icosahedron regularly tiled with pattern class p6mm

Figure 32: The icosahedron regularly tiled with pattern class p6mm

3.4. TILING THE CUBE

The *regular hexahedron*, more commonly known as the cube, consists of six square faces that meet at right angles, any of which may be regarded as the base. A total of eight vertices and twelve edges are evident, with each vertex surrounded by three square polygon faces, resulting in a vertex angle of 270 degrees (Wenninger, 1971, p.1). The cube possesses thirteen axes of rotational symmetry. Six axes of two-fold rotation pass through the centers of opposite edges and four axes of three-fold rotation coincide with the diagonals of the cube, connecting its opposite vertices. Axes of four-fold rotation also connect the midpoints of opposite faces, with all axes of rotation passing through the centre of symmetry. The existence of a centre of symmetry in the cube is shown by the fact that each face has a corresponding equal and parallel face (Shubnikov and Koptsik, 1974, p.56). The cube also exhibits a total of nine planes of reflection.

Connecting points of four-fold rotation in plane pattern classes p4, p4gm and p4mm produces a square grid, which will lend itself readily to the tiling of the cube. The application of these pattern classes to the cube maintains cubic rotational symmetry; three-fold rotation is preserved at the vertices, two-fold rotation at the mid-point of each edge and four-fold rotation at the centre of each face. A cube tiled with pattern class p4mm also retains the reflection symmetry of the cube with a total of nine reflection planes slicing through each face and each of the twelve edges. Illustrations of the cube regularly tiled with pattern classes p4, p4gm and p4mm are shown in Figures 33-34, 35-36 and 37-38 respectively.

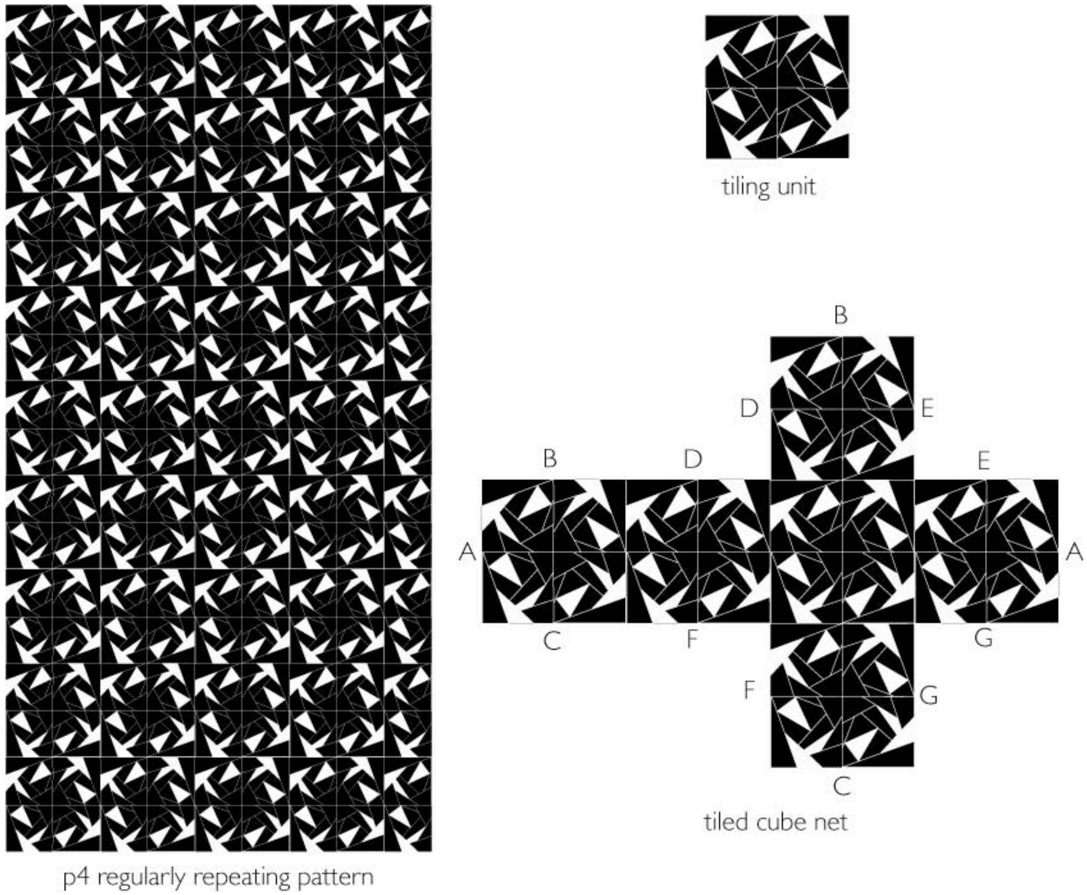

Figure 33: Illustration of a design for the cube regularly tiled with pattern class p4

Figure 34: The cube regularly tiled with pattern class p4

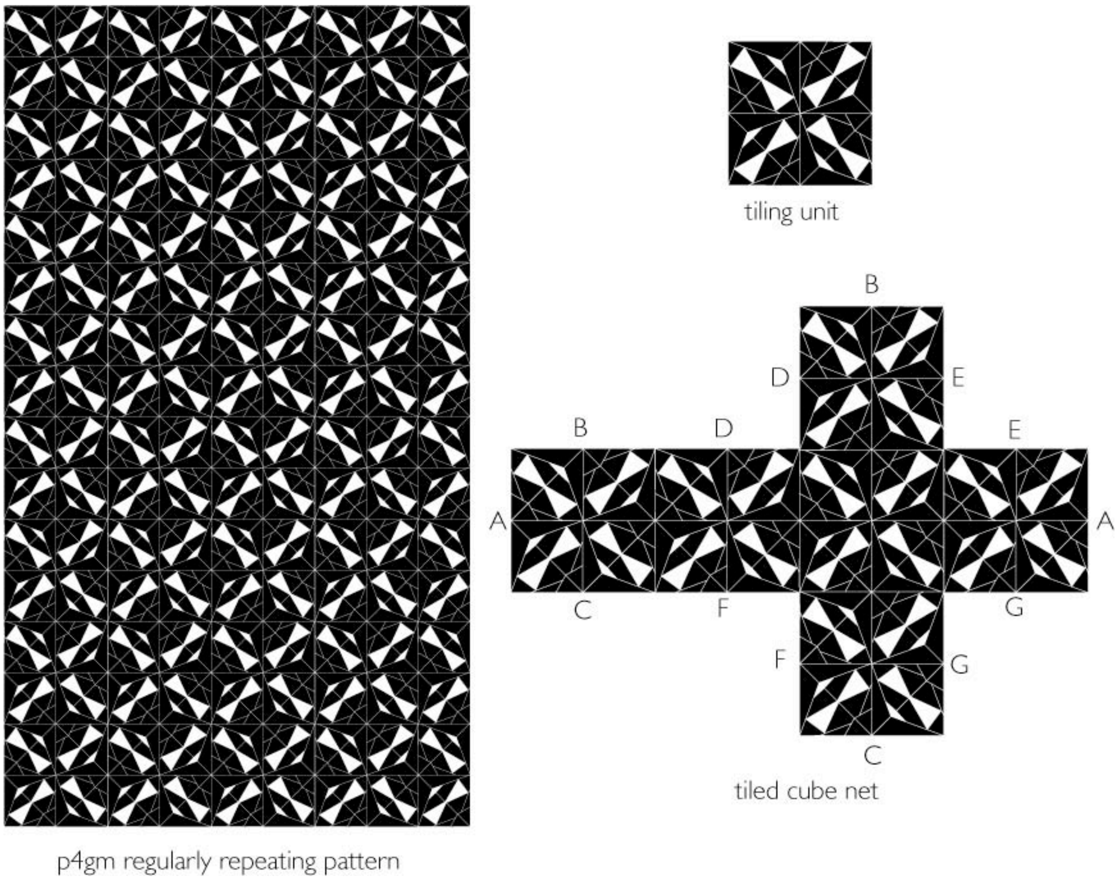

Figure 35: Illustration of a design for the cube regularly tiled with pattern class p4gm

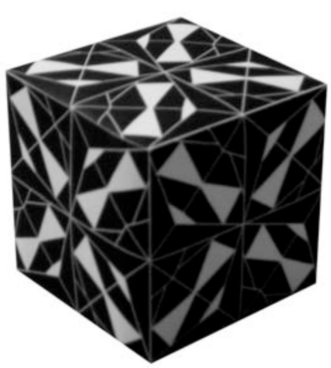

Figure 36: The cube regularly tiled with pattern class p4gm

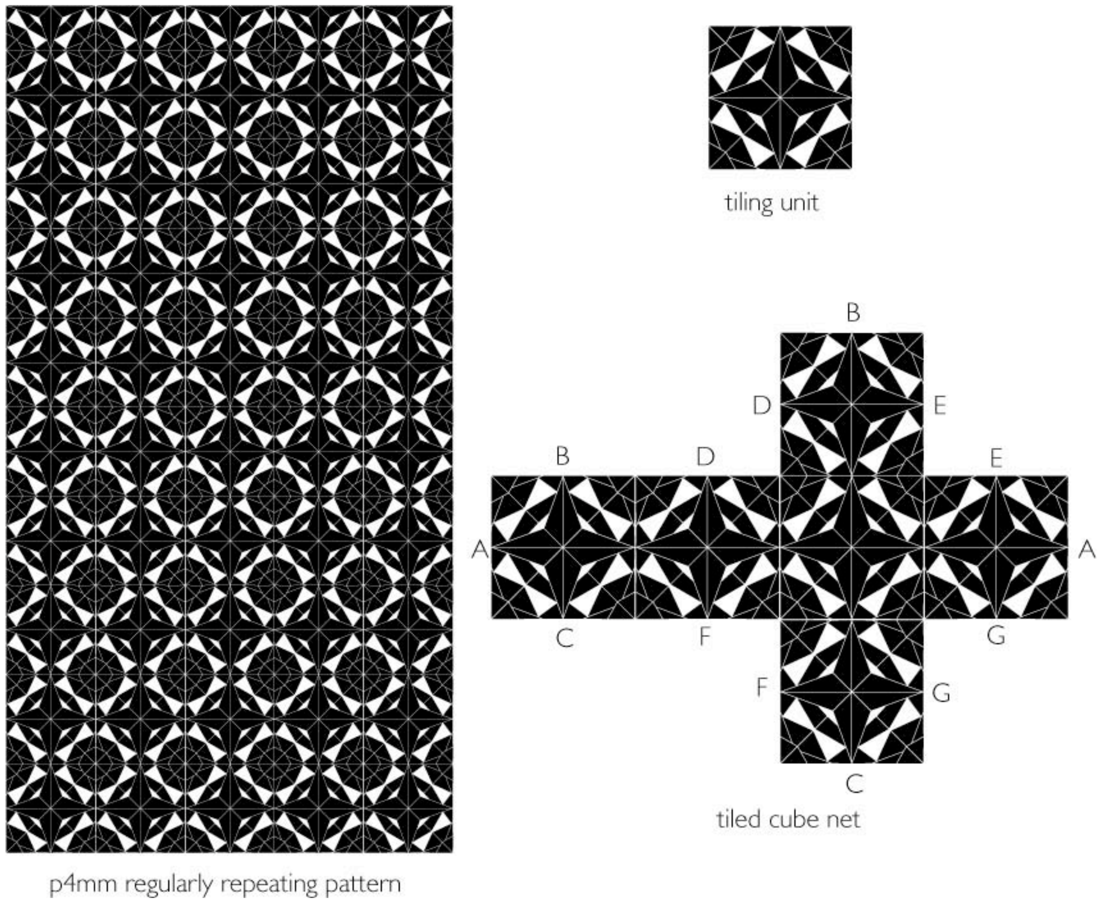

Figure 37: Illustration of a design for the cube regularly tiled with pattern class p4mm

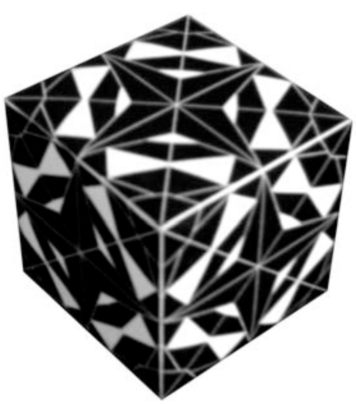
Figure 38: The cube regularly tiled with pattern class p4mm

3.5. TILING THE DODECAHEDRON

The dodecahedron is composed of twelve pentagonal faces, thirty edges and twenty vertices. Each vertex is surrounded by three pentagonal faces, displaying vertex angles of 216 degrees (Wenninger, 1971, p.1). The dodecahedron presenting thirty-one axes of rotational symmetry. Fifteen axes of two-fold rotation pass through the midpoint of opposite edges, ten axes of three-fold rotation connect opposite vertices, and six axes of five-fold rotation link the centers of opposite faces. In addition to rotational symmetry the dodecahedron also exhibits fifteen planes of reflection.

As mentioned previously, the regular tiling of the dodecahedron requires a different approach than the other Platonic solids as five-fold rotational symmetry is not possible in the seventeen all-over pattern classes. There are however fourteen types of irregular pentagons that will tessellate the plane. Of these types, the Cairo tiling (dual tessellation of the semi-regular snub square tiling 3.3.4.3.4) composed of equilateral pentagons, is geometrically the closest to a regular pentagon. The Cairo tessellation may be classified as pattern type p4gm. However, if a motif is created within each pentagonal tile that possesses five-fold rotational symmetry, the tessellation's reflection axes are destroyed and the tiling becomes classifiable as the lower symmetrical pattern class p4. A relevant illustration is given in Figure 39.

Figure 39: Illustration of the cube inscribed within the dodecahedron and the resultant tiled dodecahedron

4. IN CONCLUSION

The patterning of the five regular solids, in ways that ensure regular repetition and precise registration of the applied pattern is not a straightforward matter. When regularly repeating patterns are folded into different planes, their component parts may not correspond readily. As is the case in two-dimensions, patterns rely on their constituent symmetries to repeat by rotation (at the edges and vertices) and/or through successive reflection present at the edges of the solid. This paper identifies appropriate pattern types for each of the regular solids and presents illustrations of their application.

Only ten of the seventeen primary classes of all-over patterns can be applied to regularly repeat across the faces of the Platonic solids. These pattern classes are all based on either a hexagonal or a square lattice structure. It is apparent from Table 1 that for a plane pattern to tile the regular solids it must possess symmetry axes of two-fold rotation or higher. This contradicts an earlier study by Pawley (1962), which states that plane patterns must have a symmetry axis higher than the second order to fit onto polyhedra, as classes p2 and c2mm are suited to tiling the tetrahedron. These are the only two pattern types, discovered during this investigation, that exhibit the highest rotational symmetry of order two and are capable of tiling a Platonic solid. When applied to the tetrahedron these patterns are the only classes that do not exhibit rotation at vertices (see Figures 5 and 6). By way of summary the following rules can be presented at this stage:

- The tetrahedron, octahedron and icosahedron can only be regularly tiled with patterns based on an hexagonal lattice. The cube can only be regularly tiled with patterns based on a square lattice. Only patterns based on the hexagonal or square lattice structures are applicable to the tiling of the regular polyhedra.

- The Platonic solids can be regularly tiled by a pattern exhibiting higher rotational symmetry than the solid without altering the rotational properties of the solid. The axis of highest rotational symmetry must be present at the corners of the unit cell allowing the axes to fall on the solids vertices. Therefore, the tetrahedron, octahedron and icosahedron can be tiled with pattern classes p6 and p6mm, which contain higher rotational symmetry than the solids. The cube can be tiled with pattern classes p4, p4gm and p4mm.

- A solid can be regularly tiled with a pattern class containing lower rotational symmetry if the number of faces exhibited by the solid allows for complete repetition of the unit cells.

- The tetrahedron can be regularly tiled by a pattern possessing only rotational symmetry of the order of two provided that points of rotation are present at the mid-point of each edge. The four faces of the tetrahedron allow two complete repeats of a hexagonal unit cell across its faces. Therefore, the tetrahedron can be regularly tiled with pattern classes p2 and c2mm.

- The octahedron may be tiled with pattern classes exhibiting a highest order of rotation of three, due to the even number of faces surrounding each vertex which allow two repetitions of the unit cell around each vertices and a total of four repetitions across the solid. Therefore, the octahedron can be regularly tiled with pattern classes p3, p3m1 and p31m.

- The presence of reflection planes on a tiled solid is determined by the presence of reflection in the plane pattern.

- A solid may be regularly tiled using an area smaller than half the unit cell of a plane patterns. The tile must contain the fundamental region of the pattern and must satisfy the rotational symmetry requirements of rules two, three and four. Therefore the octahedron can be tiled with an area equal to one-sixth of the unit cell of pattern classes p3m1, p6 and p6mm.

- The dodecahedron cannot be tiled with a conventional regularly repeating pattern due to the five-fold rotational symmetry of the former. Manipulation of the Cairo tessellation (composed of equilateral pentagons) produces an appropriate tiling.

As noted previously, the research reported in this paper is a component part of a much wider ranging research project considering a wide range of geometric principles and the development of fundamental concepts relating to the use of geometry as a problem-solving tool. In the context of tiling three-dimensional solids, it is the intention to develop further this research to include the semi-regular or Archimedean solids and more complex polyhedra. The fundamental nature of this research suggests possible avenues for interdisciplinary collaborative enquiry. The relationship between solid geometry and the field of structural biology was forged nearly 50 years ago when Caspar and Klug looked outside the field of biology and drew inspiration from the geodesic domes of Buckminster Fuller in developing their theory on virus structures (Caspar and Klug, 1962). Discoveries in the fields of geometry and architecture have had a beneficial influence on virus research and although it is currently unknown how the research findings presented within this paper fit into the current body of knowledge in structural biology, there is a sense that it might prove to be a worthwhile contribution.

REFERENCES:

Caspar, D.L.D. and Klug, A. (1962). "Physical Principles in the Construction of Regular Viruses", in *Cold Spring Symp. Quant. Biol.,* XXVII, pp.1-24.Coxeter, H.S.M. (1948). *Regular Polytopes.* Methuen, London.

Coxeter, H.S.M. (1969). *Introduction to Geometry*, John Wiley and Sons, New York.

Cromwell, P.R. (1997). *Polyhedra*, Cambridge University Press, New York.

Euclid, ed. Heath, T.L. (2nd unabr. ed.) (1956). *The Thirteen Books of Euclid's Elements, Books X-XIII*, Dover, New York.

Hann, M.A. (2003). "The Fundamentals of Pattern Structure. Part I: Woods Revisited". Journal of the Textile Institute, vol.94, part 2, nos 1 and 2, pp53 - 65. "Part II: The Counter-change Challenge", Journal of the Textile Institute, vol.94, part 2, nos 1 and 2, pp66 - 80. "Part III: The Use of Symmetry Classification as an Analytical Tool", Journal of the Textile Institute, vol.94, part 2, nos. 1 and 2, pp. 81 – 88.

Hann, M. A. and Thomson G. M. (1992). The Geometry of Regular Repeating Patterns, the Textile Progress Series, vol. 22, no. 1, the Textile Institute, Manchester.

Martin, G.E. (1982). Transformation Geometry: An Introduction to Symmetry, Springer-Verlag, New York.

Pawley, G.S. (1962). "Plane Groups on Polyhedra", Acta Crystallographica, 15, pp.49-53.

Schattschneider, D. (1978). "Plane Symmetry Groups: Their Recognition And Notation", American Mathematical Monthly, 85, 6, pp.439-450.

Shubnikov, A.V. and Koptsik, V.A. (1974). Symmetry in Science and Art, Plenum Press, New York.

Stevens, P.S. (1984). Handbook Of Regular Patterns: An Introduction To Symmetry In Two Dimensions. MIT Press, Cambridge, Mass.

Washburn, D.K. and Crowe, D.W. (1988). *Symmetries of Culture: Theory and Practice of Plane Pattern Analysis*, University of Washington Press, Seattle.

Wells, D.G. (ed.) (1991). *The Penguin Dictionary of Curious and Interesting Geometry,* Penguin Books, London.

Wenninger, M.J. (1971). *Polyhedron Models.* Cambridge University Press, London.

Weyl, H. (1952). *Symmetry*, Princeton University Press, Princeton.

Woods, H.J. (1935). "The Geometrical Basis of Pattern Design. Part 3: Geometrical Symmetry in Plane Patterns", *Journal of The Textile Institute*, Transactions, 26, T341-T357.